Statistical Treatment of Light-Ray Propagation in Beam-Waveguides

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It is well known that uncorrelated, transverse displacements of the lenses of a beam-waveguide cause the light beam to deviate from its axis and that the tolerance requirements on the accuracy of the transverse lens positions are very stringent.

This paper extends the statistics of beam waveguides to include correlations between displacements of different lenses and studies the effects of a succession of uncorrelated bends.

It can be concluded from this work that the rms deviation of the light beam is proportional to the square root of the length of the waveguide if the correlation between lens displacements extends only over a limited range.

The amplitude of the Fourier component of the waveguide axis whose period equals the oscillation period of the ray has to be less than 0.2 micron if the deviation of a light beam passing through 10,000 lenses of a confocal waveguide is to be kept less than 2 mm. This requirement means that the average radius of curvature of a waveguide composed of independent circular sections of an average length of 20 m with lenses spaced 1 m apart has to be more than 10 km. The comparison between two model guides, one composed of circular section and the other of sections shaped like $\sin^2 \beta x$, *indicates that the beam deflection depends only on the average radius of curvature and average length of the sections but not on their particular shape.*

I. INTRODUCTION

Hirano, Fukatsu and Rowe[1] have studied the behavior of a light beam in a beam-waveguide whose lenses are randomly displaced from a perfectly straight line. The first two authors considered also a waveguide with sinusoidal axis displacements. The behavior of light beams in bent lens-waveguides was studied in Ref. 2. These two papers represent two extreme cases of completely uncorrelated departures of the waveguide axis from perfect straightness on the one hand and perfectly correlated departures from a straight line on the other hand.

This paper describes the statistics of a light ray by introducing a correlation function connecting different points on the waveguide axis.[7]

We show how the ray position at the nth lens depends on one Fourier component of the curvature function of the waveguide axis, while the rms value of the beam displacement depends on one frequency component of the "power spectrum" of the curvature function.

The dependence of the ray position on the Fourier component of the curvature function of the waveguide axis is analogous to the mode conversion loss of multimode waveguides.[3]

The description of the beam deflection in terms of the correlation function of the guide axis is used to draw some general conclusions. It is found that the rms value of the light beam deflection depends on the square root of the number of lenses in the guide provided that the correlation length is much less than the length of the guide. This fact can be used to deduce that the contributions of different, uncorrelated sections of the waveguide to the mean square of the beam deflection simply add up, so that this value can be computed by computing the average value of the beam deflection of one section only.

It is pointed out that apparently plausible models for the correlation function can lead to widely varying results. For this reason no attempt was made to describe the statistics of the waveguide in terms of the correlation function model.

The results of this paper can be applied to alternating gradient focusing systems since it is known[4] that the beam deviation in such systems is of the same order of magnitude as that of a system of positive lenses.

The results of this paper are of particular significance for beam waveguides composed of gas lenses since such a waveguide would use closely spaced lenses so that the number of lenses for a given length of waveguide would be very large. The tolerance requirements are proportional to the square root of the number of lenses in the guide so that the tolerances of waveguides with closely spaced gas lenses become more stringent than those of waveguides using lenses spaced far apart.

II. RELATION TO FOURIER SERIES

We use the ray description of Ref. 2 to study the statistical behavior of a light ray in a lens-waveguide.

The position of the light ray is given by its distance r_n from the lens centers. The inhomogeneous difference equation[2]

$$r_{n+2} - (2 - \kappa)r_{n+1} + r_n = Y_{n+2} \tag{1}$$

with

$$\kappa = L/f \tag{2}$$

(L = lens spacing, f = focal length) connects the ray positions at three successive lenses. The quantity Y_{n+2} at the right hand side of (1) is the distance from the center of the $(n + 2)$th lens to the point at which the straight line through the centers of the nth and $(n + 1)$th lens intersects the $(n + 2)$th lens (Fig. 1). If the lens spacing L becomes infinitesimal, $L \to 0$, Y/L^2 assumes the meaning of the inverse radius of curvature R of the waveguide axis. However, even for finite lens spacing one can define a radius of curvature R_{n+1} by the relation[2] (Fig. 1)

$$Y_{n+2} = \frac{L^2}{R_{n+1}} \tag{3}$$

so that Y_n is a measure of the curvature of the waveguide axis.

The waveguide axis can also be described by the distance S_n of the nth lens from an arbitrary straight line.[1] Between S_n and Y_n exists the following approximate relationship which can easily be derived from Fig. 1

$$Y_{n+2} = -S_{n+2} + 2S_{n+1} - S_n\,. \tag{4}$$

The solution of (1) can be given in the form[2]

$$r_n = r_n^h + r_n^i \tag{5}$$

with the solution of the homogeneous equation

$$r_n^h = r_0 \cos n\theta + \frac{r_1 - r_0 \cos \theta}{\sin \theta} \sin n\theta \tag{5a}$$

and the definition

$$\cos \theta = 1 - \tfrac{1}{2}\kappa \tag{6}$$

and the solution of the inhomogeneous equation

$$r_n^i = \frac{1}{\sin \theta} \sum_{\nu=1}^{n-1} Y_{\nu+1} \sin (n - \nu)\theta, \qquad n \geqq 2. \tag{7}$$

We will calculate the ray's departure r_n from the axis at the end of the waveguide assuming that the ray entered the waveguide on-axis, $r_0 = r_1 = 0$.

Equation (7) can be rewritten

$$r_n = \frac{1}{\sin \theta}\left\{\sin n\theta \sum_{\nu=1}^{n-1} Y_{\nu+1} \cos \nu\theta - \cos n\theta \sum_{\nu=1}^{n-1} Y_{\nu+1} \sin \nu\theta\right\}. \tag{8}$$

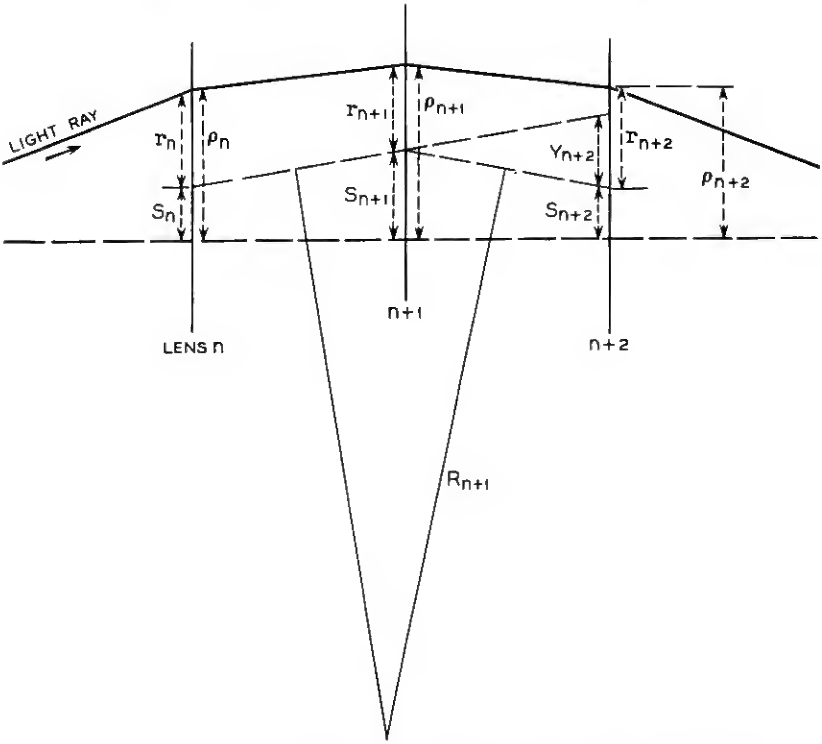

Fig. 1 — Definition of various parameters of the beam-waveguide.

The sums in (8) are Fourier coefficients as can be shown in the following way. The N-discrete points $Y_{\nu+1}$ can be represented by a Fourier series

$$Y_{\nu+1} = \sum_{\mu=-(N-1)/2}^{(N-1)/2} a_\mu \exp\left[i(2\pi/N)\mu\nu\right] \qquad \nu = 1, 2, \cdots, N. \tag{9}$$

Equation (9) is a system of N simultaneous equations for N unknown quantities a_μ. The Fourier coefficients a_μ can be calculated by multiplying (9) by $\exp\left[-i(2\pi/N)\sigma\nu\right]$ and summing over ν from 1 to N

$$a_\sigma = \frac{1}{N} \sum_{\nu=1}^{N} Y_{\nu+1} \exp\left[-i(2\pi/N)\sigma\nu\right]. \tag{10}$$

The representation (9) works if N is an odd integer. Separating a_σ into its real and imaginary part

$$a_\sigma = \alpha_\sigma + i\beta_\sigma \tag{11}$$

we obtain

$$\alpha_\sigma = \frac{1}{N} \sum_{\nu=1}^{N} Y_{\nu+1} \cos (2\pi/N)\sigma\nu \tag{12a}$$

$$\beta_\sigma = -\frac{1}{N} \sum_{\nu=1}^{N} Y_{\nu+1} \sin (2\pi/N)\sigma\nu. \tag{12b}$$

If we choose σ such $\theta = 2\pi\sigma/N$ (this may be possible only approximately but $2\pi\sigma/N$ will approximate θ closely if N is large) and denote the corresponding values of α and β by α_θ and β_θ, we get from (8)

$$r_n = \frac{n-1}{\sin\theta} \{\alpha_\theta \sin n\theta + \beta_\theta \cos n\theta\}. \tag{13}$$

This is an oscillatory function with an amplitude $r_{\max}$ which is a slowly varying function of n,

$$r_{\max} = \frac{n-1}{\sin\theta} \sqrt{\alpha_\theta^2 + \beta_\theta^2} = \frac{n-1}{\sin\theta} \mid a_\theta \mid. \tag{14}$$

The maximum deviation of the ray in the vicinity of the nth lens is determined by the magnitude of the Fourier coefficient of $Y_{\nu+1}$ belonging to the *frequency* θ. The amplitude $\mid a_\theta \mid$ has to be extremely small to keep $r_{\max}$ within reasonable limits.

If S_ν is strictly a sinusoidal function

$$S_\nu = A \sin \theta\nu \tag{15}$$

we obtain with the help of (4)

$$Y_\nu = \kappa A \sin \theta (\nu - 1) \tag{16}$$

and from (10) with $N \gg 1$

$$\mid a_\theta \mid = \tfrac{1}{2} \kappa A.$$

Equation (14) leads to

$$r_{\max} = \frac{\kappa A(n-1)}{2 \sin\theta}. \tag{17}$$

Using (3) we can also write

$$r_{\max} = \frac{L^2(n-1)}{2 R \sin\theta}$$

where R is the minimum radius of curvature.

If we take $\kappa = 2$, $L = 1$ m, $n = 10{,}000$ we obtain a waveguide of 10

km length. Requiring $r_{\max} \leqq 2$ mm we find that the amplitude A, the maximum departure of the waveguide from a straight line, has to be

$$A \leqq 0.2 \text{ micron}$$

or

$$R = 2500 \text{ km}.$$

These numbers show how extremely small the θ-Fourier component has to be!

If we take $Y_{\nu+1} = B \sin \vartheta\nu$ and substitute into (10) with $\theta = (2\pi/N)\sigma$ we get nonvanishing values for $| a_\theta |$ even if $\vartheta \neq \theta$ and $N \to \infty$. These nonvanishing values appear for all ϑ values satisfying the equation

$$\vartheta = \theta + 2\pi p \ (p = 0,1,2,3, \cdots).$$

It appears, therefore, as if we obtain Fourier components a_θ for all harmonics $\vartheta = \theta + 2\pi p$. This apparent discrepancy is resolved if we consider the period length $\lambda_\vartheta = 2\pi L/\vartheta$ of the oscillation $\sin (\vartheta/L)\nu L$. The above equation leads to the solutions for the period length

$$\lambda_\vartheta = \frac{L\lambda_\theta}{L + p\lambda_\theta} = \begin{cases} \lambda_\theta & \text{if } p = 0 \\ <L & \text{if } p \neq 0 \end{cases}$$

with $\lambda_\theta = 2\pi L/\theta$. The period length λ_ϑ is therefore either equal to λ_θ, the natural period of the ray oscillations, or it is less than L. Since the lenses are spaced a distance L apart a period of length less than L is meaningless!

III. RANDOM DISPLACEMENTS OF THE GUIDE AXIS

Our considerations so far have been limited to a definite shape of the waveguide axis. However, they can easily be extended to a statistical theory. Equation (14) can be used to obtain the rms value of the maximum beam displacement.

$$\Delta = \sqrt{\langle r_{\max}^2 \rangle} = \frac{n - 1}{\sin \theta} \sqrt{\langle | a_\theta |^2 \rangle}. \tag{18}$$

The symbol $\langle \ \rangle$ designates an ensemble average. The quantity $\langle | a_\theta |^2 \rangle$ is the expected value of the θ component of the "power spectrum" of the waveguide curvature. It is also possible to express Δ not by means of the power spectrum but by the correlation functions of Y_ν. For this purpose we write, with the help of (10),

$$\langle | a_\theta |^2 \rangle = \frac{1}{N^2} \sum_{\nu=1}^{N} \sum_{\mu=1}^{N} \langle Y_{\nu+1} Y_{\mu+1} \rangle \exp [i\theta(\mu - \nu)]. \tag{19}$$

It is reasonable to assume that $\langle Y_{\nu+1} Y_{\mu+1} \rangle$ depends only on the difference $\mu - \nu$ so that we can write

$$\langle Y_{\nu+1} Y_{\mu+1} \rangle = f_{\mu-\nu} \tag{20}$$

and to assume, furthermore,

$$f_{\mu-\nu} = f_{\nu-\mu} \,. \tag{21}$$

The factor f_λ is the correlation function of the curvature function of the waveguide axis.

Equation (19) can be rewritten in the following way:

$$\begin{aligned}\langle | a_\theta |^2 \rangle = \frac{1}{N^2} \Bigg\{ &\sum_{s=-\frac{1}{2}(N-1)}^{\frac{1}{2}(N-1)} \sum_{t=|s|+1}^{N-|s|} f_{2s} e^{2i\theta s} \\ &+ \sum_{s=1}^{\frac{1}{2}(N-1)} \sum_{t=|s|+1}^{N+1-|s|} f_{2s-1} \exp\,[i\theta(2s-1)] \\ &+ \sum_{s=-1}^{-\frac{1}{2}(N-1)} \sum_{t=|s|+1}^{N+1-|s|} f_{2s+1} \exp\,[i\theta(2s+1)] \Bigg\} .\end{aligned}$$

The summation over t can be carried out since the terms under the summation signs are independent of t. Using (21) to simplify our expression further we obtain

$$\langle | a_\theta |^2 \rangle = \frac{1}{N^2} \left\{ N f_0 + 2 \sum_{\lambda=1}^{(N-1)} (N - \lambda) f_\lambda \cos \lambda\theta \right\} \tag{22}$$

so that (18) becomes

$$\Delta = \frac{\sqrt{n-1}}{\sin\theta} \left\{ f_0 + 2 \sum_{\lambda=1}^{n-2} \left(1 - \frac{\lambda}{n-1} \right) f_\lambda \cos \lambda\theta \right\}^{\frac{1}{2}} . \tag{23}$$

If f_λ decreases with increasing λ so that the upper limit in the sum of (23) becomes immaterial the equation shows that Δ is proportional to $\sqrt{n-1}$ and not to $n - 1$ itself as one might have suspected by looking at (18). If the waveguide's curvature contains a sinusoidal component which persists throughout its length, so that no correlation length less than n exists, then the sum is proportional to its upper limit n and Δ becomes truely proportional to n itself.[1] Equation (23) expresses the rms-beam deviation in terms of the correlation function f_λ of the curvature of the waveguide axis. It is easy to rewrite this equation as an expression depending on the correlation function of the waveguide axis displacement itself. The displacement of the nth lens from a straight line is S_n . Defining

$$\langle S_\nu S_\mu \rangle = G_{\nu-\mu} = G_{\mu-\nu} \tag{24}$$

we get with the help of (4) and (20),

$$f_\lambda = 6G_\lambda - 4G_{\lambda-1} - 4G_{\lambda+1} + G_{\lambda-2} + G_{\lambda+2}. \tag{25}$$

Substituting (25) into (23), rearranging terms leads to

$$\Delta = \frac{\sqrt{n-1}\,\kappa}{\sin\theta}\left\{G_0 + 2\sum_{\lambda=1}^{\infty}[1-\lambda/(n-1)]G_\lambda\cos\lambda\theta \right.$$
$$+ \frac{2}{(n-1)\kappa^2}\left[2(1+\kappa-\tfrac{1}{2}\kappa^2)G_0 - (2-\kappa)G_1\right.$$
$$\left.\left. - 4\kappa\sqrt{\kappa-\tfrac{1}{4}\kappa^2}\sum_{\lambda=1}^{\infty}G_\lambda\sin\lambda\theta\right]\right\}^{\frac{1}{2}}$$

where we assumed that the correlation length $\Lambda \ll n$. If we assume that $n \gg 1$ (26) simplifies

$$\Delta \approx \frac{\sqrt{n}\,\kappa}{\sin\theta}\left\{G_0 + 2\sum_{\lambda=1}^{\infty}G_\lambda\cos\lambda\theta\right\}^{\frac{1}{2}}. \tag{27}$$

This approximate equation shows that Δ is very nearly proportional to the θ component of the Fourier coefficient of the correlation function.

If the lens displacements S_ν are uncorrelated, $G_\lambda = 0$ for $\lambda \neq 0$, we have

$$\Delta = 2\sqrt{\frac{\kappa}{4-\kappa}}\,\delta\sqrt{n} \tag{28}$$

with

$$\delta = \sqrt{G_0} = \sqrt{\langle S_n^2\rangle}.$$

Equation (28) differs from Ref. 1 (17) (for large values of n) by a factor of $\sqrt{2}$. The reason for the occurrence of this additional factor in our theory is that our Δ is the rms value for the amplitude of the oscillatory beam trajectory while (17), Ref. 1 describes the rms value of all points of the oscillatory beam trajectory.

Let us assume that the waveguide axis is composed of sections of a given average length and that the curvature function of one section is uncorrelated to that of any of the other sections. All sections are assumed to be of the same type. For example, they may all be circular bends which differ only in length and radius of curvature. A waveguide of this type has a finite correlation length and if n, the total number of lenses, is

large we get from (23)

$$\Delta_n^2 = An \tag{30}$$

with

$$A = \left(\frac{1}{\sin\theta}\right)^2 \left\{f_0 + 2\sum_{\lambda=1}^{\infty} f_\lambda \cos\lambda\theta\right\}. \tag{31}$$

Adding one more section with m lenses changes n to $n + m$ in (30). The increase of Δ^2 due to the addition of a section with m lenses is given by $\Delta_m^2 = Am$ so that we get

$$\Delta_{n+m}^2 = \Delta_n^2 + \Delta_m^2 \tag{32}$$

for a guide with $n + m$ lenses.

Each section of the guide can be thought of as such an additional section so that

$$\Delta^2 = \sum_{\nu=1}^{M} \Delta_{m_\nu}^2$$

with $\Delta_{m_\nu}^2$ being the contribution of the νth section and M the number of sections. Introducing the average value $\Delta_{\mathbf{m}}^2$, we obtain

$$\Delta = \Delta_{\mathbf{m}}\sqrt{M}. \tag{33}$$

The rms beam deviation of a waveguide composed of sections can be obtained by calculating the contribution of the rms beam deviation of each section, computing their rms values and applying (33).

IV. EXAMPLES

As a first example we consider a waveguide composed of circular sections which are connected so that the first derivative is continuous. The departure of the light beam which goes through the center of the first two lenses of one of the circular arcs is given by (7)

$$r_n = Y\,\frac{\sin n\dfrac{\theta}{2}\sin\,(n-1)\,\dfrac{\theta}{2}}{\sin\theta\,\sin\dfrac{\theta}{2}} \tag{34}$$

with $Y = Y_\nu = \text{const}$. If the circular sections have an average number of $\mathbf{m}$ lenses which vary with a Gaussian distribution with variance σ_m we get with the help of (33)

$$\Delta = \frac{\sqrt{\frac{1}{2}M\langle Y^2\rangle}}{\sin\theta\,\sin\frac{\theta}{2}}\left\{1 + \tfrac{1}{2}\cos\theta - 2\cos\frac{\theta}{2}\cos\left(\mathbf{m} - \tfrac{1}{2}\right)\theta\exp\left(\tfrac{1}{2}\theta^2\sigma_m^{\,2}\right) + \tfrac{1}{2}\cos\left(2\mathbf{m} - 1\right)\theta\exp\left(-2\theta^2\sigma_m^{\,2}\right)\right\}^{\frac{1}{2}} \tag{35}$$

with M circular sections per waveguide. It was assumed that Y and m are statistically independent.

The validity of (35) was checked by a computer simulated *experiment*. We constructed 30 different waveguides composed of a series of circular arcs. The quantities Y and m were computed as Gaussian random variables with mean value $\langle Y \rangle = 0$ and $\langle m \rangle = \mathbf{m}$. Each waveguide contained 10,000 lenses. One light ray was traced through each guide with the use of (1) and the rms value of the values r_n with n = 10,000 was computed which, multiplied by $\sqrt{2}$, should equal the value Δ of (35). This experiment was repeated three times with different values of $\mathbf{m}$. In all experiments we considered the confocal case, $\kappa = 2$, $\cos\theta = 0$.

Table I shows how (35) compares to the computer results. The σ_m values in Table I were chosen for the convenience of the computer calculations.

The agreement between the theoretical and *experimental* values is quite good considering that the rms value was computed from only 30 samples.

We can relate $\langle Y^2 \rangle$ to an average radius of curvature $\mathbf{R}$ since according to (3)

$$\langle Y^2\rangle = L^4\left\langle\frac{1}{R^2}\right\rangle = \frac{L^4}{\mathbf{R}^2}. \tag{36}$$

Table I

n = 10,000

m	σ_m	$\frac{\Delta}{\sqrt{\langle Y^2\rangle}}$ (equation (35))	$\frac{\Delta}{\sqrt{\langle Y^2\rangle}}$ Computer Experiment
3	0.92	77.9	60.2
20	6.1	22.4	21.9
100	30.0	10.0	8.7

TABLE II

m	R
3	39 km
20	11.2 km
100	5.0 km

We may allow $\Delta = 0.2$ cm at the end of the waveguide of 10 km length ($L = 1m$). The permissable values of **R**, computed from the theoretical values of Table I, are listed in Table II.

As a second example we consider a guide which is built up of sections of tapered bends formed according to Fig. 2

$$S_\nu = A \sin^2 (\pi/m)(\nu - 1) \qquad \nu = 1,2\cdots m. \tag{37}$$

The amplitudes A and the number of lenses m are random variables. Substituting (37) into (4) and (7) we obtain

$$r_{n+1} = -A \frac{\sin^2 \dfrac{\pi}{m}}{\sin\theta} \cdot \frac{\left(\sin\theta n\right)\left(\cos\dfrac{2\pi}{m} - \cos\theta\right) + \left(\sin\theta\right)\left(\cos\dfrac{2\pi}{m} n - \cos\theta n\right)}{2\sin\left(\dfrac{\theta}{2} - \dfrac{\pi}{m}\right)\sin\left(\dfrac{\theta}{2} + \dfrac{\pi}{m}\right)}. \tag{38}$$

The lens numbered $n = m + 1$ is the last lens on the bend. According to (5a) the amplitude of the oscillations is given by

$$r_{\max}^2 = r_0^2 + \left(\frac{r_1 - r_0\cos\theta}{\sin\theta}\right)^2. \tag{39}$$

We restrict ourselves to the confocal case $\kappa = 2$, $\cos\theta = 0$ and set $r_0 = r_m$, $r_1 = r_{m+1}$ because the oscillation caused by the bend is taken as the

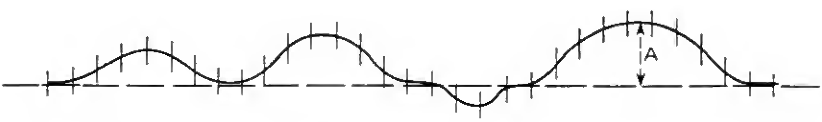

Fig. 2 — Beam-waveguide composed of sections of tapered bends.

initial condition of the ray. We obtain from (38) and (39)

$$r_{\max}{}^2 = A^2 \frac{\left(\sin^4 \frac{\pi}{m}\right)\left(1 + \cos^2 \frac{2\pi}{m}\right)\left(1 - \cos \frac{\pi}{2} m\right)}{2\left[\sin\left(\frac{1}{4} - \frac{1}{m}\right)\pi \sin\left(\frac{1}{4} + \frac{1}{m}\right)\pi\right]^2}. \tag{40}$$

To compute Δ we must average $r_{\max}{}^2$ over A^2 and m. We assume that A and m are uncorrelated and that m follows a Gaussian distribution with average value $\mathbf{m}$ and variance σ_m. To simplify the averaging π/m is replaced by $\pi/\mathbf{m}$ and these terms are taken out of the integral. This procedure is valid only if $\pi/\mathbf{m} \ll 1$ so that we may also take $\sin \pi/\mathbf{m} = \pi/\mathbf{m}$ and $\cos 2\pi/\mathbf{m} = 1$. Remembering that $M = n/\mathbf{m}$, we obtain from (33)

$$\frac{\Delta}{\sqrt{\langle A^2 \rangle}} = \frac{\pi^2 \sqrt{M} \sqrt{1 - \cos \frac{\pi}{2} \mathbf{m} \exp\left(-\frac{1}{8}\pi^2 \sigma_m{}^2\right)}}{\mathbf{m}^2 \cdot \sin\left(\frac{1}{4} - \frac{1}{\mathbf{m}}\right)\pi \cdot \sin\left(\frac{1}{4} + \frac{1}{\mathbf{m}}\right)\pi} \tag{41}$$

with n being the total number of lenses of the guide and $\mathbf{m}$ the average number of lenses per section.

The equation (41) is valid if $\mathbf{m} \gg 1$ and $\sigma_m/\mathbf{m} \ll 1$. It shows that Δ decreases as the number of lenses per bend is increased.

Some numerical results are shown in Table III. The column labeled "computer result" again contains data calculated by tracing rays through simulated beam-waveguides. The waveguides were "constructed" of arcs according to (37) with A and m being Gaussian random variables. Thirty random waveguides were constructed for each value of $\mathbf{m}$ and the rms values of the ray position r_n with $n = 10{,}000$ was computed from these 30 samples.

The computer results agree with the theoretical values to the order of magnitude. The agreement in this second example is poorer than that of Table I. However, the 30 values of $r_{10{,}000}$ of this example scatter more widely than those of the first example. Omitting the largest of the 30 values for $\mathbf{m} = 50$ changes Δ by a factor of 0.65 which shows that the statistics of these 30 samples is not very reliable. By comparison omitting the largest value of the sample with $\mathbf{m} = 20$ of the first example changes Δ only by a factor of 0.95.

The results of Table III can be used to get an impression of the line tolerances required for a nominally straight waveguide. Let us assume that the 10,000 lenses of our model guide are spaced 1 m apart ($L = 1m$)

TABLE III

$n = 10{,}000$			
m	σ_m	$\frac{\Delta}{\sqrt{\langle A^2 \rangle}}$ equation (41)	Computer Result
10	2.8	7.70	12.6
50	15	1.12 10^{-1}	8.8 10^{-1}
250	77	2.00 10^{-3}	2.7 10^{-3}

which results in a guide 10 km long. If this is a guide built of gas lenses of 0.65 cm diameter (about $\frac{1}{4}$ in.) we might allow $\Delta = 0.2$ cm to be reasonably sure that the light beam will get through the pipe. Fixing Δ allows us to calculate the rms value of the amplitudes $\sqrt{\langle A^2 \rangle}$ of the deviation from straightness and the average radius of curvature of the guide. An average radius of curvature $\mathbf{R}$ at the peak of the arc of (37) is given by

$$\frac{1}{\mathbf{R}} = \left\langle \left| \frac{1}{R} \right| \right\rangle \approx \frac{2\pi^2 \sqrt{\langle A^2 \rangle}}{\mathrm{m}^2 L^2} \approx 2\Delta \frac{\sqrt{\mathrm{m}}}{\sqrt{n}\, L^2} \sin\left(\frac{1}{4} - \frac{1}{\mathrm{m}}\right)\pi \sin\left(\frac{1}{4} + \frac{1}{\mathrm{m}}\right)\pi. \tag{42}$$

The second half of this equation was obtained by substituting (41) for $\langle A^2 \rangle^{\frac{1}{2}}$ assuming that $\frac{1}{8}\pi^2\sigma_m^2 \gg 1$. Table IV shows the values of the rms amplitudes and the average permissable radius of curvature which were computed from the theoretical values of Table III. The tolerance requirements are rather stringent as Table IV shows. For short bends, departures from straightness of only a fraction of a millimeter can be allowed while this tolerance moves up into the meter range as the average length of the bend exceeds hundreds of meters. In case of uncorrelated random wiggles our present example of $\Delta = 2$ mm and $n = 10{,}000$ lenses leads to an rms value for the position tolerances of $\delta = 0.01$ mm according to (28). This is a very real tolerance requirement

TABLE IV

mL	$\sqrt{\langle A^2 \rangle}$	$\mathbf{R}$
10 m	2.6 10^{-2} cm	19.4 km
50 m	1.78 cm	7.1 km
250 m	1.00 10^{2} cm	3.16 km

since there will always be random lens displacements superimposed on longer bends so that the uncorrelated, random component of lens displacements has to be kept below 10 microns.

A comparison of Table II with Table IV shows that the average radii of curvature permissable for our two examples are nearly the same. This is true even though the waveguides of the two examples are of very different construction. This may support the belief that the average radius of curvature and the length of the sections determine the deflection of the ray regardless of how the waveguide is shaped in detail. The agreement between the values of Table II and IV is improved if one corrects for the different length of the sections.

It may be in order to add a remark concerning models for the correlation function G_λ. Since the correlation has to be of finite length one might be tempted to try a correlation function of the form[6]

$$G_\lambda = G_0 \exp\left(-\frac{|\lambda|}{q}\right) \tag{43}$$

with q being the number of lenses within the correlation distance. The correlation number q must be of the same order of magnitude as the average number $\mathbf{m}$ of lenses per section of our model waveguide. Substituting (43) into (27) we obtain

$$\Delta = \frac{\sqrt{n}\,\kappa}{\sin\theta}\sqrt{G_0} \cdot \sqrt{1 - 2\exp(-1/q)\frac{\exp(-1/q) - \cos\theta}{1 + \exp(-2/q) - 2\exp(-1/q)\cos\theta}} \tag{44}$$

or if $1/q \ll 1$

$$\Delta = \frac{\sqrt{n}\,\kappa}{\sin\theta}\sqrt{\frac{G_0}{q(1 - \cos\theta)}}. \tag{45}$$

This correlation function is obviously a poor model for a waveguide with random bends since Δ of (45) decreases with q only like $q^{-\frac{1}{2}}$ while Δ of (41) decreases like $\mathbf{m}^{-2}$. For $q = 250$ we obtain from (45), setting $\sqrt{G_o} \approx \sqrt{\langle A^2\rangle}$ ($G_0 \approx \frac{1}{2}\sqrt{\langle A^2\rangle}$ for the arcs of (37)),

$$\frac{\Delta}{\sqrt{\langle A^2\rangle}} = 6.4$$

which is three orders of magnitude larger than corresponding values of Table III.

Another possible choice for a correlation function may be

$$G_\lambda = G_0 \exp(-\lambda^2/q^2). \tag{46}$$

Using the identity[5]

$$\sum_{\lambda=-\infty}^{\infty} \exp(-\lambda^2/q^2) \cos \lambda\theta = \sqrt{\pi}\, q \sum_{\nu=-\infty}^{\infty} \exp(-\pi^2 q^2[(\theta/2\pi) - \nu]^2) \quad (47)$$

we obtain from (27)

$$\frac{\Delta}{G_0} = \frac{\pi^{\frac{1}{4}}\kappa\sqrt{n}}{\sin\theta}\sqrt{q}\,\{\exp\{-\pi^2 q^2[(\theta/2\pi) - 1]^2\} + \exp(-\tfrac{1}{4}q^2\theta^2) + \exp\{-\pi^2 q^2[(\theta/2\pi) + 1]^2\}\}^{\frac{1}{2}} \quad (48)$$

where all but three terms of the sum on the right hand side of (47) are neglected which is justified if $q > 1$.

The maximum rms beam deviation of (48) decreases with increasing q like

$$\sqrt{q}\exp(-\tfrac{1}{8}q^2\theta^2)$$

that is much faster than (45). These two examples indicate how critically Δ depends on the shape of the correlation function. It appears that more insight can be gained by choosing models for the random deviation of the waveguide curvature rather than by trying to guess at model correlation functions.

The reason for the critical dependence of Δ on the shape of the correlation function can be seen from the following argument.

In our second example, (37) we obtain $G_0 \approx 0.5\langle A^2\rangle$. However the ratio $\Delta/\sqrt{\langle A^2\rangle}$ is, for example, in the order of 0.1 according to the second line of Table III. From (27) we obtain, with $\kappa = 2$ and $n = 10^4$,

$$\frac{\Delta}{\sqrt{\langle A^2\rangle}} \approx 2\; 10^2 \sqrt{0.5 + 2\sum_{\nu=1}^{\infty} \frac{G_\lambda}{\langle A^2\rangle}\cos\lambda\theta.}$$

If $\Delta/\sqrt{\langle A^2\rangle}$ is to be 0.1, the sum under the square root sign must be very nearly equal to -0.25 so that the two terms under the square root sign cancel to a term of the order of magnitude 10^{-6}. A very slight variation of the value of the sum gives rise to a large variation of Δ.

As a last example we consider a waveguide with M random tilts. If each tilt is located at a lens, Ref. 2 (31), gives for the beam amplitude caused by one tilt with angle α

$$r_{\max} = 2\alpha \frac{L}{\sqrt{4\kappa - \kappa^2}} \quad (49)$$

so that (33) leads to

$$\Delta = 2\sqrt{\langle\alpha^2\rangle}\frac{L}{\sqrt{4\kappa - \kappa^2}}\sqrt{M}. \quad (50)$$

If $n = 10{,}000$, $L = 1m$, $\kappa = 2$, and $M = 100$ we find that

$$\sqrt{\langle\alpha^2\rangle} \leqq 2\ 10^{-4} \text{ radians} = 0.0115°$$

if $\Delta \leqq 2$ mm is required.

V. CONCLUSION

Is it more advantageous to space the lenses of a beam-waveguide closely or farther apart? The answer to this question depends on our ability to control tolerances. Equation (41) shows that the rms beam deviation due to random bends decreases rapidly with increasing number of lenses, while (28) indicates that the rms beam deviation due to uncorrelated lens displacements increases slowly with increasing lens number. Only practical experience can tell how to compromise between these two conflicting requirements.

APPENDIX

Equivalence of Two Representations

The problem of ray propagation in a bent lens-waveguide has been treated in Ref. 2 and by Hirano and Fukatsu.[1] The treatments of the problem in these two papers differ in the way the ray is described. In Ref. 2 the ray position r_n is measured from the center of the lenses and the position of the lenses with respect to each other is described by a quantity Y_n (Fig. 1). In Ref. 1, a straight reference line is used to determine the position ρ_n of the ray as well as the lens displacements S_n. The ray position r_n at the nth lens in the representation of Ref. 2 is given by

$$r_n = \frac{1}{\sin\theta} \sum_{\nu=1}^{n-1} Y_{\nu+1} \sin (n-\nu)\theta \qquad n \geqq 2. \tag{51}$$

The values of r_n at $n = 0$ and $n = 1$ are $r_0 = r_1 = 0$.

In the representation of Ref. 1 the solution of the inhomogeneous difference equation reads

$$\rho_n = \frac{\kappa}{\sin\theta} \sum_{\nu=1}^{n-1} S_\nu \sin (n-\nu)\theta \qquad n \geqq 2 \tag{52}$$

with $\rho_0 = \rho_1 = 0$.

With the help of Fig. 1 it is easy to see that approximately

$$S_{n+2} = 2S_{n+1} - S_n - Y_{n+2}\sqrt{1 - (S_{n+1} - S_n)^2/L^2}, \tag{53}$$

or if

$$\frac{S_{n+1} - S_n}{L} \ll 1$$

$$Y_{n+2} = 2S_{n+1} - S_{n+2} - S_n\,, \tag{53a}$$

and

$$\rho_n = r_n + S_n\,. \tag{54}$$

The substitution of (53a) and (54) into (51) leads to

$$\begin{aligned}\rho_m &= S_n + \frac{1}{\sin\theta}\sum_{\nu=1}^{n-1}(2S_\nu - S_{\nu+1} - S_{\nu-1})\sin(n-\nu)\theta \\ &= S_n + \frac{1}{\sin\theta}\Big\{S_1\sin(n-1)\theta - S_n\sin\theta - S_0\sin(n-1)\theta \\ &\quad + \sum_{\nu=1}^{n-1} S_\nu[2\sin(n-\nu)\theta - \sin(n-\nu+1)\theta - \sin(n-\nu-1)\theta]\Big\}\end{aligned}$$

which can be simplified to

$$\begin{aligned}\rho_n = (S_0 - S_1)\cos n\theta - (S_0 - S_1)\cos\theta\sin n\theta \\ + \frac{\kappa}{\sin\theta}\sum_{\nu=1}^{n-1} S_\nu\sin(n-\nu)\theta.\end{aligned} \tag{55}$$

The first two terms with $S_0 - S_1$ are solutions of the homogeneous difference equation and appear here because $r_0 = r_1 = 0$ does not coincide with $\rho_0 = \rho_1 = 0$ if S_0 and S_1 are unequal to zero.

Since the first two terms can always be removed by adding a suitable solution of the homogeneous equation the equivalence of (51) and (52) has been shown.

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